6.3 Ecological models

predator-prey system (Lotka-Volterra eqs)

$x(t)$: population of prey (rabbits)

$y(t)$: " " predator (wolves)

$$\frac{dx}{dt} = ax - pxy \qquad a, b, p, q > 0$$

$$\frac{dy}{dt} = -by + qxy$$

what do they say?

if there is no y, then $\frac{dx}{dt} = ax$ exponentially growing

if there is no x, then $\frac{dy}{dt} = -by$ declines exponentially

$$\frac{dx}{dt} = x(a - py)$$ wolves reduce rabbits' growth rate

$$\frac{dy}{dt} = y(-b + qx)$$ rabbits boosting wolves growth rate

critical pts: $\underbrace{(0,0)}_{\text{both die out}}$, $\overbrace{\left(\frac{b}{q}, \frac{a}{p}\right)}^{\text{both at stable pop.}}$

Jacobian: $J(x,y) = \begin{bmatrix} a - py & -px \\ qy & -b + qx \end{bmatrix}$

goal: what happens to x, y as $t \to \infty$

<u>example</u>

$$\frac{dx}{dt} = x - 0.5xy = x(1 - 0.5y)$$

$$\frac{dy}{dt} = -0.75y + 0.25xy = y(-0.75 + 0.25x)$$

cp: $(0,0)$, $(3, 2)$

as $t \to \infty$, what happens to x? to y?

$$J = \begin{bmatrix} 1-0.5y & -0.5x \\ 0.25y & -0.75+0.25x \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -0.75 \end{bmatrix} \qquad \lambda = 1, -0.75$$

saddle
unstable
not sensitive to perturbation

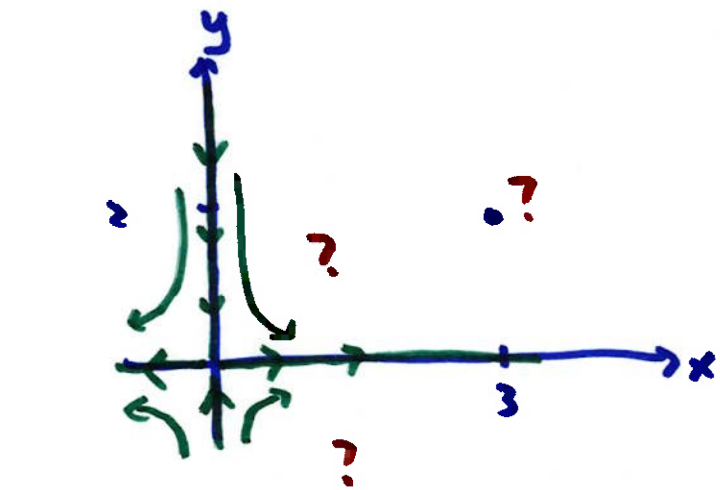
$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

solution curves near (0,0)

$$\vec{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-0.75t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

suppose initial condition is such that $c_1 = 0$ then solution goes to (0,0) along y-axis

if $c_2 = 0$, solutions leave (0,0) along x-axis



if initial x is not zero, then as $t \to \infty$, we do NOT see both pop. go to zero.

$$J(3,2) = \begin{bmatrix} 0 & -1.5 \\ 0.5 & 0 \end{bmatrix}$$

$\lambda = \pm \frac{\sqrt{3}}{2} i$ pure imaginary

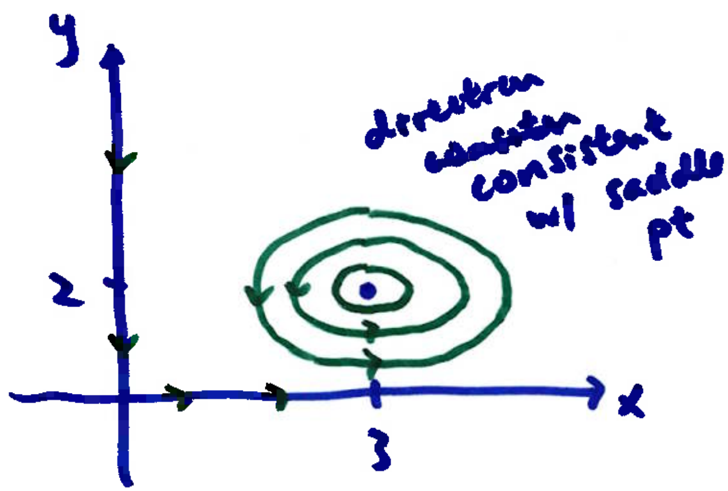
center

stable

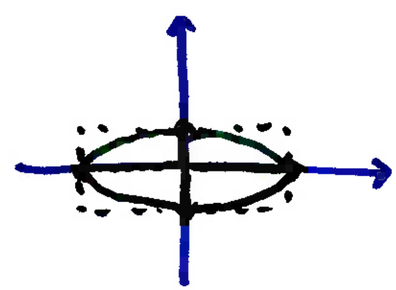
sensitive to perturbation

(subject to verification)

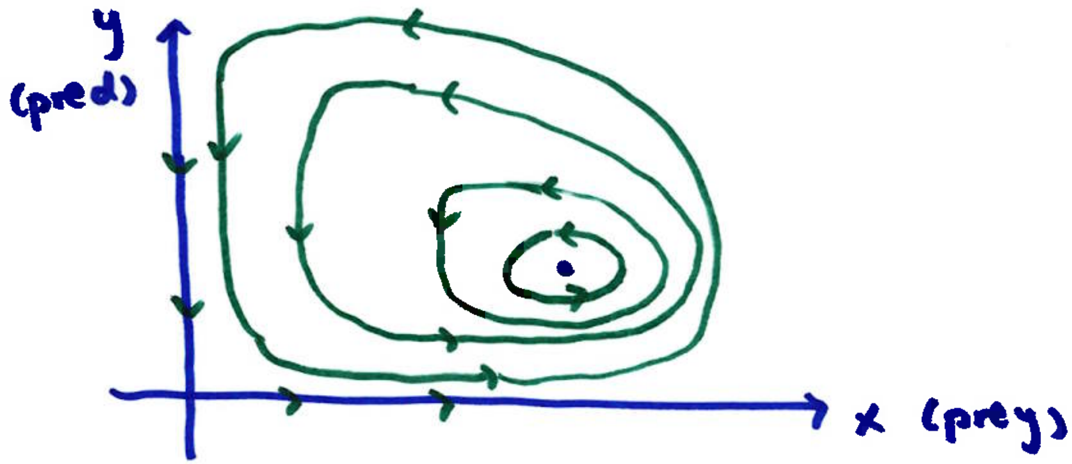
ellipses centred at (3, 2)



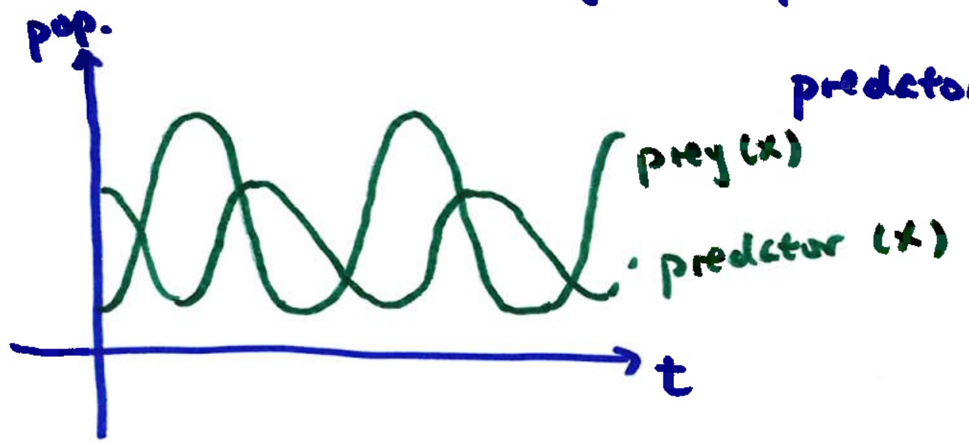
$$\vec{v} = \begin{bmatrix} 1 \\ \pm \frac{1}{\sqrt{3}} i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$



more complete picture



as x increases, more food is available to y, so y increases but as y increases, it eats more x so x eventually declines which means less food for y, so y declines which then allow x to grow and the cycle repeats.



predator lags behind prey

same period for both : $\frac{2\pi}{\sqrt{ab}}$ independent of initial condition

phase shift (lag) is $\frac{1}{4}$ of that

amplitude of x (prey) is $\frac{Kb}{\delta}$ K: some # depending on initial condition

" " y (predator) is $\frac{K\sqrt{ab}}{p}$

finally, to verify the cp (3,2) we plot nonlinear phase diagram or solve

$$x' = x(a-py)$$
$$y' = y(-b+\delta x)$$
$$\vdots$$
$$\frac{dy}{dx} = \frac{y(-b+\delta x)}{x(a-by)}$$
$$\vdots$$
$$a\ln y - py + b\ln x - \delta x = C$$

warped ellipses